Basics of Magnetohydrodynamics (MHD)

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What is MHD?

- MHD = fluid description of plasmas

Plasma physics

- Ionized gases
  - Conductivity $\sigma > 0$
- Macroscopically neutral (if $\mathbf{E}_{\text{ext}} = 0$):
  - Debye shielding generates Yukawa potentials:

$$\Phi(r) = -\frac{q}{r} \exp\left(-\sqrt{Z+1}\frac{r}{\lambda_D}\right); \text{ where } \lambda_D := \sqrt{\frac{k_B T}{4\pi \bar{n}_e e^2}}.$$

  - Coulomb interaction effectively short-ranged
- Magnetic fields $\mathbf{B}$ from small drift velocities $v_{\text{drift}} = |v_e - v_i|$
  - Can be quite large for high σ

Hydrodynamics

- Fluid description:
  - Many random, short-ranged interactions
  - Scale hierarchy: $\lambda \ll L \ll L_{\text{sys}}$
  - Collisional invariants: N, p, E

    Kinetic theory:
    Liouville/Boltzmann eq.:
    $$\frac{\mathrm{d}f}{\mathrm{d}t}(t, \mathbf{x}, \mathbf{p}) \;=\; C[f]$$

  - Macroscopic conservation laws fixing the fluid variables $\rho, P, \mathbf{v}$

MHD = Hydrodynamics applied to plasma

- Main difference to neutral gas: Coulomb interaction
  - Electrons & ions form two separate, coupled fluids
  - *But:* Single-fluid behavior for negligible charge separation:

    $\lambda_D \ll L$
- $\mathbf{B}$-field adds contrib. $\mathcal{M}_{ij}$ to the fluid stress-energy tensor T_{ij}
  (or: Lorentz force into Vlasov equation)

MHD = Hydrodynamics applied to plasma

- Dynamical variables and evolution laws in MHD:
  - ρ: Continuity equation
  - $\mathbf{v}$: Euler (or Navier-Stokes) equation (+ Lorentz force)
  - P: Energy conservation equation (+ Ohmic heat gain term)
  - $\mathbf{B}$: Maxwell's equations

The induction equation

- Assumptions:
  - Non-relativistic plasma (linearize in $|\mathbf{v}|/c$)
  - High, spatially constant conductivity ($\sigma \gg T^{-1}$)
  - Ohm's law
- Evolution of $\mathbf{B}$-field then follows from Maxwell's equations:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi q & \qquad \boldsymbol{\nabla} \times \mathbf{E} &= -\partial_{ct}\mathbf{B} \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 & \qquad \boldsymbol{\nabla} \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j} + \underbrace{\partial_{ct}\mathbf{E}}_{\sim\mathcal{O}(|v|^2/c^2)\cdot\mathcal{O}(\frac{B}{L})}, \end{aligned}$$

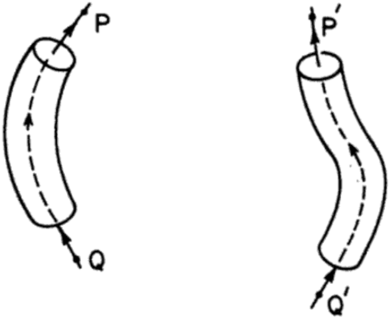
- and implies the induction equation:

The induction equation

- Induction equation:
$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\frac{c^2}{4\pi\sigma}\Delta\mathbf{B}}_{\text{diffusion}} + \underbrace{\boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B})}_{\text{advection}}$$
- Magnetic Reynolds number:
$$\mathcal{R}_M := \frac{\text{advection}}{\text{diffusion}} \sim \frac{v|\mathbf{B}|}{L} \Big/ \frac{c^2}{4\pi\sigma}\frac{|\mathbf{B}|}{L^2} = \frac{4\pi\sigma L v}{c^2}$$
  - Diffusivity limit: $\mathcal{R}_M \ll 1$ (typical for lab situations)
  - Ideal MHD limit: $\mathcal{R}_M \gg 1$ (typical for astrophys. systems)

Ideal MHD limit

- Perfect conductivity: $\sigma \to \infty$
  - $\dfrac{\partial \mathbf{B}}{\partial t} \approx \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B})$
  - Field lines "frozen" into plasma (Alfvén's theorem of flux-freezing)
  - *i.e.* fluid elements connected by field line remain connected:



(Source: Choudhuri: The Physics of Fluids and Plasmas)

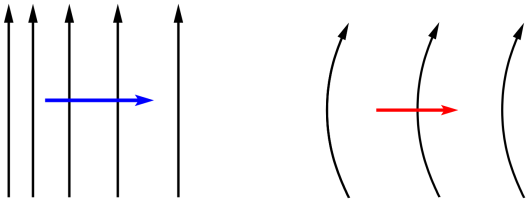
Magnetic field backreaction on plasma

- Euler equation modified by Lorentz force:

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla P + \frac{1}{c}(\mathbf{j}\times\mathbf{B})$$

$$\underset{\text{Ampère's law}}{=} -\nabla\left(P + \underbrace{\frac{\mathbf{B}^2}{8\pi}}_{\text{Mag. pressure}}\right) + \underbrace{\frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B}}_{\text{Mag. tension}}$$

- $\Rightarrow$ Field lines resist to being squeezed or bent:



(Source: Nick Murphy, Lecture at Harvard-Smithsonian Center for Astrophysics)

Summary

- MHD $=$ Dynamic theory of magnetized fluids
- Single-fluid approximation for tight Coulomb coupling
  - Induction equation fixes $\mathbf{B}$
    - Ideal MHD limit $\sigma \to \infty$: Advection only (Flux-freezing)
  - Backreaction of $\mathbf{B}$ on fluid quantities $\rho, \mathbf{v}, P$ via Lorentz-force